Stochastic Calculus for Option Pricing (Script)

Stochastip

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This is a script for the YouTube channel "Stochastip"

Acknowledgment

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1 Stochastic Processes

In this section, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.1 (Stochastic Process)**.** *Any family $X = \{X_t, t \geq 0\}$ of random variables taking values in $\mathbb{R}^d$ is called a continuous-time stochastic process, or simply a stochastic process.*
A process X can be seen as a family of random variables indexed by time:

$$X_t : \omega \in (\Omega, \mathcal{F}) \mapsto X_t(\omega) \in (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)).$$

Here, $\{t \geq 0\} = \mathbb{R}^+$ represents the time, Ω is the sample space, $\mathcal{B}(\mathbb{R}^+)$ is the Borel sigma-algebra on $\mathbb{R}^+$, and $\mathcal{F}$ is a sigma-algebra on Ω. The process takes values in $\mathbb{R}^d$, which is equipped with the Borel sigma-algebra $\mathcal{B}(\mathbb{R}^d)$.

We can then be interested in the distributions of the random variables X_t:

Definition 1.2. *Let $X = \{X_t, t \geq 0\}$ be a stochastic process.*

- ***[Stationary process]*** *A process X is said to be stationary if for any $h \geq 0$, the process $X_{\cdot+h} := \{X_{t+h}, t \geq 0\}$ is equivalent to X.*
- ***[Stationary increments - Independent increments]*** *A process X is said to have stationary increments if for any t, the distribution of $X_{t+h} - X_t$ does not depend on t. The process X is said to have independent increments if for any $n \geq 1$ and any $0 = t_0 \leq t_1 \leq \cdots \leq t_n \in \mathbb{R}^+$, the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.*

- ***[Sample path of a process]*** *For any $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is called the sample path of X (associated with the state ω). The process is said to have continuous sample paths if for any $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is continuous on $\mathbb{R}^+$.*

- ***[Measurable process]*** *A process $X = \{X_t\}_{t\geq 0}$ is said to be measurable if the mapping $(t, \omega) \mapsto X_t(\omega)$ is measurable when defined on the product space $\mathbb{R}^+ \times \Omega$, which is equipped with the product sigma-algebra $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$.*

  This means that for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$, the set

  $$\{(t, \omega) : X_t(\omega) \in A\}$$

  belongs to $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$.

2 Filtrations and Measurability

A stochastic process X models the evolution over time of a random quantity. Observing the evolution of X over time conveys information: the past realizations of X and its current realization can inform about its future evolution. The mathematical tool that allows us to translate this idea of evolving information is the notion of filtration.

Definition 2.1. *Let $X = (X_t)_{t\geq 0}$ be a stochastic process.*

- ***[Filtration]*** *A filtration is a family $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ of sigma-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s \leq t$. If the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a filtration $\mathcal{F}$, we refer to it as the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{F})$.*

- ***[Adapted process]*** *We say that $X = \{X_t, t \geq 0\}$ is $\mathcal{F}$-adapted if for all $t \geq 0$, the random variable X_t is $\mathcal{F}_t$-measurable.*

3 Gaussian Processes

This paragraph is dedicated to a particular type of process: Gaussian processes. Before defining these processes, we start with a brief review of Gaussian vectors.

Proposition 3.1. *Let X be a Gaussian random variable with $\mathbb{E}[X] = \mu$ and Var$(X) = \sigma^2$, then its characteristic function is given by:*

$$\Phi_X(\xi) = \mathbb{E}[e^{i\xi X}] = \exp\left(i\xi\mu - \frac{\xi^2\sigma^2}{2}\right),$$

Proof. Let X be a Gaussian random variable with mean $\mathbb{E}[X] = \mu$ and variance $\mathrm{Var}(X) = \sigma^2$. The characteristic function of X is:

$$\begin{aligned}
\Phi_X(\xi) &= \mathbb{E}[e^{i\xi X}] \\
&= \int_{-\infty}^{\infty} e^{i\xi x} f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} e^{i\xi x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(i\xi x - \frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{-2\sigma^2 i\xi x + x^2 + \mu^2 - 2x\mu}{2\sigma^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2 - 2(\mu + i\xi\sigma^2)x + \mu^2)}{2\sigma^2}\right) dx \\
&= \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-(\mu+i\xi\sigma^2))^2}{2\sigma^2}\right)}_{=1} \exp\left(-\frac{-(\mu+i\xi\sigma^2)^2 + \mu^2}{2\sigma^2}\right) dx \\
&= \exp\left(-\frac{-\mu^2 + \xi^2\sigma^4 - 2\sigma^2 i\xi\mu + \mu^2}{2\sigma^2}\right) \\
&= \exp\left(-\frac{\xi^2\sigma^2}{2}\right) \exp\left(i\xi\mu\right) \\
&= \exp\left(i\xi\mu - \frac{\xi^2\sigma^2}{2}\right).
\end{aligned}$$

Thus, the characteristic function of a Gaussian random variable X is:

$$\Phi_X(\xi) = \exp\left(i\xi\mu - \frac{\xi^2\sigma^2}{2}\right).$$

□

Definition 3.2 (Gaussian Vector). *Let $X = (X_1, \cdots, X_N)$ be a random vector defined on $(\Omega, \mathcal{F}, \mathbb{P})$. It is said to be Gaussian if and only if every linear combination of its components is a Gaussian random variable. That is, for any $a \in \mathbb{R}^N$, the random variable $\langle a, X\rangle = \sum_{k=1}^N a_k X_k$ is Gaussian.*

Proposition 3.3. *Let $X_1, \cdots, X_N$ be independent Gaussian random variables. Then the vector $(X_1, \cdots, X_N)$ is a Gaussian vector.*

Proof. The proof uses the injectivity of the characteristic function (see section A.2 for details). According to the definition of a Gaussian vector, it is sufficient to show that the random variable $\langle a, X\rangle = a^T X = \sum_{k=1}^N a_k X_k$ is a Gaussian random variable for any $a \in \mathbb{R}^N$. Let $m_k = \mathbb{E}[X_k]$, $\sigma_k^2 = \mathrm{var}(X_k)$. Then

$$\Phi_{\langle a,X\rangle}(\xi) = \mathbb{E}\left[e^{i\xi\langle a,X\rangle}\right] = \mathbb{E}\left[e^{i\xi\sum_{k=1}^N a_k X_k}\right] = \mathbb{E}\left[\prod_{k=1}^N e^{i\xi a_k X_k}\right] = \prod_{k=1}^N \mathbb{E}\left[e^{i\xi a_k X_k}\right]$$

Since X_k are independent,

$$= \prod_{k=1}^{N} \exp\left(i\xi a_k m_k - \frac{\xi^2 \sigma_k^2 a_k^2}{2}\right) = \exp\left(i\xi \sum_{k=1}^{N} a_k m_k - \frac{\xi^2 \sum_{k=1}^{N} \sigma_k^2 a_k^2}{2}\right)$$

which is the characteristic function of a Gaussian random variable with mean $\left(\sum_{k=1}^{N} a_k m_k\right)$ and variance $\left(\sum_{k=1}^{N} \sigma_k^2 a_k^2\right)$. By the injectivity of the characteristic function, $\langle a, X\rangle$ is a Gaussian random variable. □

Proposition 3.4. *If $X := (X_1, \cdots, X_N)$ is a Gaussian vector, then its characteristic function is:*

$$\Phi_X : \xi \in \mathbb{R}^N \mapsto \exp\left(i\langle \xi, \mathbb{E}[X]\rangle - \frac{1}{2}\langle \xi, \Sigma\xi\rangle\right)$$

where Σ is the covariance matrix of X.

Proof. According to the definition of a Gaussian vector, for any $\xi \in \mathbb{R}^N$, the random variable $\sum_{k=1}^{N} \xi_k X_k$ is a Gaussian random variable. Therefore,

$$\begin{aligned}
\Phi_{\langle \xi, X\rangle}(1) &= \mathbb{E}[e^{i\langle \xi, X\rangle}] = \mathbb{E}\left[e^{i\sum_{k=1}^{N} \xi_k X_k}\right] \\
&= \exp\left(i \cdot \mathbb{E}\left[\sum_{k=1}^{N} \xi_k X_k\right] - \frac{1}{2}\text{var}\left(\sum_{k=1}^{N} \xi_k X_k\right)\right) \\
&= \exp\left(i\,\langle \xi, \mathbb{E}[X]\rangle - \frac{1}{2}\sum_{k,l=1}^{N} \xi_k \xi_l \text{cov}(X_k, X_l)\right) \\
&= \exp\left(i\,\langle \xi, \mathbb{E}[X]\rangle - \frac{1}{2}\,\langle \xi, \Sigma\xi\rangle\right).
\end{aligned}$$

□

Theorem 3.5. *Let $X := (X_1, \cdots, X_N)$ be a Gaussian vector. Its components are independent if and only if they are uncorrelated.*

Proof. The proof of "independence implies zero correlation" is immediate. Now suppose that the components X_k are uncorrelated. Then the matrix Σ is diagonal. By the previous proposition,

$$\Phi_X(\xi) = \exp\left(i\,\langle \xi, \mathbb{E}[X]\rangle - \frac{1}{2}\,\langle \xi, \Sigma\xi\rangle\right)$$

Since Σ is diagonal,

$$\begin{aligned}
&= \exp\left(i\sum_{k=1}^{N} \xi_k \mathbb{E}[X_k] - \frac{1}{2}\sum_{k=1}^{N} \xi_k^2 \text{var}(X_k)\right) \\
&= \prod_{k=1}^{N} \exp\left(i\xi_k \mathbb{E}[X_k] - \frac{1}{2}\xi_k^2 \text{var}(X_k)\right)
\end{aligned}$$

$$= \prod_{k=1}^{N} \Phi_{X_k}(\xi_k),$$

which implies, via Proposition A.1 on page 188, the independence of the components of the vector X. □

Definition 3.6 (Gaussian Process)**.** *Let $X := \{X_t, t \geq 0\}$ be a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process X is said to be Gaussian if for all $N \in \mathbb{N}$ and for all $t_0 \leq \cdots \leq t_N \in \mathbb{R}^+$, the vector $(X_{t_0}, \cdots, X_{t_N})$ is Gaussian.*
If X is a Gaussian process, then (by Proposition 2.2) its law is completely characterized by its mean function $e_X : t \mapsto \mathbb{E}[X_t]$ and its covariance operator $\Sigma : (s,t) \mapsto cov(X_s, X_t)$. Some examples:

- *The Brownian motion starting from zero (to be studied later) is a Gaussian process $B = \{B_t, t \geq 0\}$ with continuous sample paths, whose mean function e_B and covariance operator K_B are given by $e_B(t) = 0$ and $K_B(s,t) = \min(s,t)$.*
- *The Brownian bridge is a Gaussian process $\Pi = \{\Pi_t, t \in [0,1]\}$ with continuous sample paths satisfying $e_\Pi(t) = 0$ and $K_\Pi(t,s) = \min(s,t) - st$.*

4 Martingales in Continuous Time

Definition 4.1 ($\mathcal{F}$-martingale in continuous time)**.** *Let $M = \{M_t, t \geq 0\}$ be an adapted stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathcal{F}$. We say that M is a martingale if:*

1. *For all $t \geq 0$, the random variable M_t is integrable;*
2. *For all $0 \leq s \leq t$, $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$.*

When, instead of point 2, the process M_t satisfies $\forall s \leq t : M_s \leq \mathbb{E}[M_t|\mathcal{F}_s]$ (resp. $\forall s \leq t : M_s \geq \mathbb{E}[M_t|\mathcal{F}_s]$), the process M is called a sub-martingale (resp. super-martingale).

Intuition 2.2.1 If we consider a game where one gains M_t at any instant t and we compare the expected gain at time t given that at the present time $s \leq t$ the gain is M_s, then a martingale is a fair game, a super-martingale is a losing game, and a sub-martingale is a winning game.

Definition 4.2 (Stopping time)**.** *Let $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ be a filtration. A random variable $\tau : \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ is a stopping time if for all $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$.*

Definition 4.3 (Local $\mathcal{F}$-martingale in continuous time)**.** *Let $M = \{M_t, t \geq 0\}$ be an adapted stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathcal{F}$. The process M is said to be a local martingale if there exists a strictly increasing sequence of stopping times $(T_n)_{n \geq 1}$ such that:*

1. *The sequence T_n converges almost surely to $+\infty$;*

2. For all $n \geq 0$, the process $M^{T_n} := \{M_{t\wedge T_n}, t \geq 0\}$ is a martingale.

Proposition 4.4. *Let $M = \{M_t, t \geq 0\}$ be a martingale defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathcal{F}$. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a convex function such that for all $t \geq 0$, the random variable $\Phi(M_t)$ is integrable. We define the process M^Φ by: $M^\Phi_t := \Phi(M_t)$, $t \geq 0$.*

1. *M^Φ is a sub-martingale on $(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{F})$;*
2. *If M is only a sub-martingale but Φ is increasing, then M^Φ is also a sub-martingale.*

Proof. One needs to use Jensen's inequality for conditional expectations (see section A.5, Proposition A.5, point 9 on page 192).

□

The result of Proposition A.7 on page 194, valid for discrete-time sub-martingales, can be extended to sub-martingales (and in particular to martingales) with continuous paths. In fact, it is sufficient to have processes with càdlàg paths (right-continuous with left limits, see [15]). We obtain:

Proposition 4.5 (Doob's maximal inequality). *Let $M = \{M_t, t \geq 0\}$ be a sub-martingale defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathcal{F}$. Suppose M has continuous paths. Then for all $0 \leq \tau < T$ and all $\lambda > 0$, we have*

$$\lambda \mathbb{P}\left(\sup_{s\in[\tau,T]} M_s \geq \lambda\right) \leq \mathbb{E}[|M_T| 1_{\{\sup_{s\in[\tau,T]} M_s \geq \lambda\}}] \leq \mathbb{E}[|M_T|].$$

Important technique 2.2.1 The result is demonstrated in the appendix for the discrete-time case. To pass to continuous time, we will use the (countable) set of rational times and use continuity for the other times.

Proof. Let D be the countable set $D := ([\tau, T] \cap \mathbb{Q}) \cup \{T\}$.

1. First, we show that $\sup_{s\in[\tau,T]} M_s = \sup_{s\in D} M_s$. Since $D \subset [\tau, T]$, it is clear that $\sup_{s\in[\tau,T]} M_s \geq \sup_{s\in D} M_s$. Let $t \in [\tau, T]$ and $\omega \in \Omega$ be arbitrary but fixed. We can find a sequence $s_n \in D$ such that $s_n \to t$. Then $\sup_{s\in D} M_s(\omega) \geq M_{s_n}(\omega)$ for all n. By taking the limit and using the continuity of the path, we obtain $\sup_{s\in D} M_s(\omega) \geq M_t(\omega)$. Since this holds for arbitrary t and ω, we get $\sup_{s\in D} M_s(\omega) \geq \sup_{t\in[\tau,T]} M_t(\omega)$, leading to the announced equality.
2. It is possible to consider a sequence of finite sets $(F_n)_{n\geq 0}$ such that $F_n \subset F_{n+1} \subset D$, $T = \max F_n$ for all n, and $\bigcup_n F_n = D$. Proposition A.7 applied to the sub-martingale $\{M_s, s \in F_n\}$ gives:

$$\lambda \mathbb{P}\left(\sup_{s\in F_n} M_s \geq \lambda\right) \leq \mathbb{E}[|M_T| 1_{\{\sup_{s\in F_n} M_s \geq \lambda\}}] \leq \mathbb{E}[|M_T| 1_{\{\sup_{s\in[\tau,T]} M_s \geq \lambda\}}].$$

   Since the sequence of sets F_n is increasing:

$$\mathbb{P}\left(\sup_{s\in D} M_s \geq \lambda\right) = \lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in F_n} M_s \geq \lambda\right).$$

We then conclude that:

$$\lambda\mathbb{P}\left(\sup_{s\in D} M_s \geq \lambda\right) = \lim_{n\to\infty} \lambda\mathbb{P}\left(\sup_{s\in F_n} M_s \geq \lambda\right) \leq \mathbb{E}[|M_T|1_{\{\sup_{s\in[\tau,T]} M_s\geq\lambda\}}].$$

□

The following result is a consequence of Proposition 2.4.

Proposition 4.6 (Doob's inequality II). *Let M be a martingale defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathcal{F}$. Suppose M has continuous paths. Then for all $0 \leq \tau < T$ and $p > 1$, we have*

$$\mathbb{E}\left[\left(\sup_{s\in[\tau,T]} |M_s|\right)^p\right] \leq \frac{p^p}{(p-1)^p}\mathbb{E}[|M_T|^p].$$

Proof. Let $M^* := \sup_{s\in[\tau,T]} |M_s|$. Let L be a positive real number. Then

$$\mathbb{E}[(M^* \wedge L)^p] = \int_0^L p\lambda^{p-1}\mathbb{P}(M^* \geq \lambda)d\lambda = \int_0^L p\lambda^{p-1}1_{\{M^*\geq\lambda\}}d\lambda.$$

By Fubini's theorem, we have

$$\mathbb{E}[(M^* \wedge L)^p] = \int_0^L p\lambda^{p-1}\mathbb{P}(M^* \geq \lambda)d\lambda.$$

Applying Proposition 2.4 to the sub-martingale $|M|$, we get:

$$\mathbb{E}[(M^* \wedge L)^p] \leq \int_0^L p\lambda^{p-2}\mathbb{E}[|M_T|1_{\{M^*\geq\lambda\}}]d\lambda = \int_0^L p\lambda^{p-2}\mathbb{E}[|M_T|1_{\{M^*\geq\lambda\}}]d\lambda.$$

Applying Hölder's inequality, we get:

$$\mathbb{E}[(M^* \wedge L)^p] \leq (\mathbb{E}[|M_T|^p])^{\frac{1}{p}} (\mathbb{E}[(M^* \wedge L)^p])^{\frac{p-1}{p}}.$$

It follows that:

$$\mathbb{E}[(M^* \wedge L)^p] \leq \frac{p^p}{(p-1)^p}\mathbb{E}[|M_T|^p].$$

Finally, letting $L \to +\infty$ and using the monotone convergence argument (see Proposition A.2, point A.2 on page 190), we obtain the desired inequality.

□

Intuition 2.2.2 The previous inequalities are useful to obtain information about the behavior of the entire trajectory of a sub-martingale $\{M_t, t \geq 0\}$ using only the values of M_T at the final time T.

5 Brownian Motion

Definition 5.1 (Real Brownian Motion)**.** *Let $W = W_t, t \geq 0$ be a process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:*

1. *W is a process with continuous paths;*

2. *W has independent increments, i.e., for all $0 \leq s \leq t$, the random variable $W_t - W_s$ is independent of $\mathcal{F}_s$;*

3. *For all $0 \leq s \leq t$, the random variable $W_t - W_s$ follows a normal distribution with mean 0 and variance $t - s$.*

If additionally $W_0 = 0$, we say that W is a standard Brownian motion (respectively non-standard Brownian motion). When the filtration $\mathbb{F}$ is given a priori, a process $\mathbb{F}$-adapted that satisfies the above conditions is called an $\mathbb{F}$-Brownian motion. In all that follows, unless explicitly stated otherwise, we will assume $W_0 = 0$.

5.1 Remarks

There is a lot of information in this definition that needs to be detailed.

1. The state of the system at time t, W_t, is a Gaussian random variable with mean 0 ($\mathbb{E}[W_t] = 0$) and variance $\mathbb{E}[W_t^2] = t$, which increases the longer the system evolves.

2. The probability that W_t belongs to a small interval $[x, x+dx]$ is given by the Gaussian density:

$$\mathbb{P}(W_t \in [x, x + dx]) = g(t, x)dx = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx.$$

   In particular, with at least 95% probability, $|W_t| \leq 2\sqrt{t}$ (see Figure 1.1). This does not exclude the possibility of a large movement in a short time, but it is unlikely. Such bounds play a significant role in finance.

3. The random variable W_t is the sum of its increments, i.e., the sum of independent Gaussian random variables with the same distribution. This infinitesimal decomposition is the basis of stochastic differential calculus.

4. This property is a consequence of a classic result about Gaussian variables with distribution $N(m, \sigma^2)$:

$$\mathbb{P}(|X - m| \leq 2\sigma) \approx 0.95.$$

6 Properties of Brownian Motion

Proposition 6.1 (Properties of Brownian Motion)**.**

1. **Symmetry Properties:** If $\{W_t; t \geq 0\}$ is a standard Brownian motion, then $\{-W_t; t \geq 0\}$ is also a standard Brownian motion.

2. **Scaling Properties:** If $\{W_t; t \geq 0\}$ is a Brownian motion, then for any $c > 0$, the process $\{W_{ct}; t \geq 0\}$ defined by $\left\{\frac{1}{\sqrt{c}} W_{ct}; t \geq 0\right\}$ is also a Brownian motion.

3. **Time Reversal:** The time-reversed process $W^r = \{W_T - W_{T-t}; t \in [0, T]\}$ is a Brownian motion on $[0, T]$.

4. **Time Inversion:** The time-inverted process $\left\{\frac{1}{t} W_{1/t}; t > 0, W_0 = 0\right\}$ is a Brownian motion.

Proof:

. 1. **Symmetry Properties:**

   Let $\{W_t; t \geq 0\}$ be a standard Brownian motion. We need to show that $\{-W_t; t \geq 0\}$ is also a standard Brownian motion. We check the three defining properties of a standard Brownian motion:

   (a) *Continuous paths:* Since W_t has continuous paths and the negative of a continuous function is also continuous, $\{-W_t; t \geq 0\}$ has continuous paths.

   (b) *Independent increments:* For any $0 \leq s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$. Since $-W_t + W_s = -(W_t - W_s)$, and the negative of an independent increment is still an independent increment, $\{-W_t; t \geq 0\}$ has independent increments.

   (c) *Normal distribution of increments:* For any $0 \leq s \leq t$, $W_t - W_s \sim N(0, t - s)$. Thus, $-W_t + W_s \sim N(0, t - s)$ since multiplying by -1 does not change the mean or variance of a normal distribution. Therefore, $-W_t + W_s \sim N(0, t - s)$.

   Hence, $\{-W_t; t \geq 0\}$ is also a standard Brownian motion.

2. **Scaling Properties:**

   Let $\{W_t; t \geq 0\}$ be a standard Brownian motion. We need to show that the process $\left\{\frac{1}{\sqrt{c}} W_{ct}; t \geq 0\right\}$ is also a standard Brownian motion. We check the defining properties:

   (a) *Continuous paths:* Since W_t has continuous paths, W_{ct} has continuous paths for any $c > 0$. Multiplying by a constant $\frac{1}{\sqrt{c}}$ does not affect continuity, so $\left\{\frac{1}{\sqrt{c}} W_{ct}; t \geq 0\right\}$ has continuous paths.

   (b) *Independent increments:* For any $0 \leq s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$. Hence, $W_{ct} - W_{cs}$ is independent of $\mathcal{F}_{cs}$ because $\mathcal{F}_{cs} \subseteq \mathcal{F}_{ct}$. Multiplying by $\frac{1}{\sqrt{c}}$ does not change the independence of increments.

   (c) *Normal distribution of increments:* For any $0 \leq s \leq t$, $W_t - W_s \sim N(0, t - s)$. Then, $W_{ct} - W_{cs} \sim N(0, c(t - s))$. Hence, $\frac{1}{\sqrt{c}}(W_{ct} - W_{cs}) \sim N(0, t - s)$ because scaling a normal random variable by $\frac{1}{\sqrt{c}}$ scales the variance by $\frac{1}{c}$.

   Therefore, $\left\{\frac{1}{\sqrt{c}} W_{ct}; t \geq 0\right\}$ is also a standard Brownian motion.

3. **Time Reversal:**

   Let $\{W_t; t \geq 0\}$ be a standard Brownian motion. We need to show that $W^r = \{W_T - W_{T-t}; t \in [0, T]\}$ is a Brownian motion on $[0, T]$. We check the properties:

(a) *Continuous paths:* W_t has continuous paths, so $W_T - W_{T-t}$ has continuous paths since it is composed of continuous functions.

(b) *Independent increments:* For any $0 \leq s \leq t \leq T$, consider the increments $W^r_{t_2} - W^r_{t_1} = (W_T - W_{T-t_2}) - (W_T - W_{T-t_1}) = W_{T-t_1} - W_{T-t_2}$. Since W_t has independent increments, $W_{T-t_1} - W_{T-t_2}$ is independent of the past increments of the form $W_{T-s} - W_{T-u}$ for $u < s \leq t_1$. Thus, W^r has independent increments.

(c) *Normal distribution of increments:* For any $0 \leq s \leq t \leq T$, the increment $W_{T-s} - W_{T-t} \sim N(0, t-s)$. Thus, $W^r_{t_2} - W^r_{t_1} \sim N(0, t_1 - t_2)$.

Hence, $W^r = \{W_T - W_{T-t}; t \in [0, T]\}$ is a Brownian motion on $[0, T]$.

4. **Time Inversion:**

Let $\{W_t; t \geq 0\}$ be a standard Brownian motion. We need to show that the process $\left\{\frac{1}{t}W_{1/t}; t > 0, W_0 = 0\right\}$ is a Brownian motion. We check the properties:

(a) *Continuous paths:* W_t has continuous paths. The mapping $t \mapsto \frac{1}{t}$ is continuous for $t > 0$, and $W_{1/t}$ has continuous paths. Multiplying by $\frac{1}{t}$ does not affect continuity, so $\left\{\frac{1}{t}W_{1/t}; t > 0\right\}$ has continuous paths.

(b) *Independent increments:* Consider $0 < s < t$. The increments of the time-inverted process are given by $\frac{1}{s}W_{1/s} - \frac{1}{t}W_{1/t}$. For $0 < s < t$, $W_{1/s} - W_{1/t}$ is independent of $\mathcal{F}_{1/t}$. Scaling by $\frac{1}{s}$ and $\frac{1}{t}$ does not change the independence of these increments.

(c) *Normal distribution of increments:* The increments $\frac{1}{s}W_{1/s} - \frac{1}{t}W_{1/t}$ are normal because $W_{1/s} - W_{1/t} \sim N(0, 1/t - 1/s)$. Scaling by $\frac{1}{s}$ and $\frac{1}{t}$, we get $\left(\frac{1}{s} - \frac{1}{t}\right)\frac{W_{1/s} - W_{1/t}}{\sqrt{1/t - 1/s}} \sim N(0, 1/t - 1/s)$.

Hence, $\left\{\frac{1}{t}W_{1/t}; t > 0, W_0 = 0\right\}$ is a Brownian motion.

□